

PROJECTED RICHARDSON VARIETIES AND AFFINE SCHUBERT VARIETIES

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ABSTRACT. Let G be a complex simple algebraic group and G/P be a partial flag variety. The projections of Richardson varieties from the full flag variety form a stratification of G/P . We show that the closure partial order of projected Richardson varieties agrees with that of a subset of Schubert varieties in the affine flag variety of G . Furthermore, we compare the torus-equivariant cohomology and K -theory classes of these two stratifications by pushing or pulling these classes to the affine Grassmannian. Our work generalizes results of Knutson, Lam, and Speyer for the Grassmannian of type A .

1. INTRODUCTION

Let G be a complex simple algebraic group, $B, B^- \subset G$ be opposite Borel subgroups, and $T = B \cap B^-$ the maximal torus. The flag variety G/B has a stratification $G/B = \cup X_w = \cup X^w$ by Schubert varieties $X_w = \overline{B^- w B / B}$ and opposite Schubert varieties $X^w = \overline{B w B / B}$. The intersections $X_w^v = X_w \cap X^v$ are known as Richardson varieties, and also form a stratification of G/B . Let $P \subset B$ be a fixed parabolic subgroup, and $\pi : G/B \rightarrow G/P$ denote the projection. The *projected Richardson varieties* $\Pi_w^v = \pi(X_w^v)$ form a stratification of G/P which was studied by Lusztig [27] and Rietsch [32] in the context of total positivity, and by Goodearl and Yakimov [10] in the context of Poisson geometry. Projected Richardson varieties enjoy many desirable geometric properties: Knutson, Lam, and Speyer [17] (see also Billey and Coskun [1]) showed that they are Cohen-Macaulay, normal, have rational singularities and are exactly the compatibly Frobenius split subvarieties of G/P with respect to the standard splitting.

Our main results are combinatorial, cohomological, and K -theoretic comparisons between the projected Richardson varieties Π_w^v and the affine Schubert varieties of the affine flag variety \widetilde{Fl} of G . These results generalize work of Knutson, Lam, and Speyer [16] in the case that G/P is a Grassmannian of type A . The techniques of our proof differ significantly from those of [16]. In particular, the proof of the cohomological part of [16] appears to only extend to cominuscule G/P . A more geometric comparison in the Grassmannian case was performed by Snider [33] who also recovered our K -theoretic comparison in the case of the Grassmannian.

On the combinatorial side, we compare two posets. One is obtained from the closure order of projected Richardson varieties, which we denote by Q_J . It was first studied by Rietsch [32] and Goodearl and Yakimov [10]. The other one is the *admissible subset* $\text{Adm}(\lambda)$ associated to a dominant coweight λ , introduced by Kottwitz and Rapoport in [20]. It is a subset of the extended affine Weyl group \widehat{W} and the Bruhat order on \widehat{W} gives a partial order on $\text{Adm}(\lambda)$. One important result in the study of Shimura varieties is that the special fiber of the local model is a union of finitely many opposite affine Schubert

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cells IwI/I in the affine flag variety, where w runs over the admissible set $\text{Adm}(\lambda)$ for the Shimura coweight λ . See [29] and [37].

In this paper, we define an injection $\theta : Q_J \rightarrow \text{Adm}(\lambda)$. Our combinatorial theorem states that θ is order-reversing and the image is the $W \times W$ -double coset of the translation element $t^{-\lambda}$. In the special case where λ is a cominuscule coweight, θ gives an order-reversing bijection between Q_J and $\text{Adm}(\lambda)$. The proof relies on properties of the *Demazure*, or *monoidal* product of Coxeter groups, studied for example by He, and He and Lu in [12, 13, 14]. In Section 3, we describe some applications of our result to the combinatorial properties of the poset Q_J , and to the enumeration of $\text{Adm}(\lambda)$ for cominuscule coweights λ . We also give a closed formula for the number of rational points of the special fiber of local model for “fake” unitary Shimura varieties.

In fact, our combinatorial theorem naturally extends to the comparison of a larger partial order on $W \times W^J \supset Q^J$ with a $W \times W$ -double coset in \widehat{W} . This partial order on $W \times W^J$ arises as the closure partial order of a stratified space Z_J , studied by Lusztig [28]. In Section 4.1 we give maps between these stratified spaces which in part explains the combinatorial theorems.

Now let $\text{Gr} = G(\mathbf{K})/G(\mathbf{O})$ denote the the affine Grassmannian of G , where $\mathbf{K} = \mathbb{C}((t))$ and $\mathbf{O} = \mathbb{C}[[t]]$. Let $\text{Gr}_\lambda \subset \text{Gr}$ denote the closure of the $G(\mathbf{O})$ -orbit containing the torus-fixed point $t^{-\lambda}$ labeled by $-\lambda$. The dense open orbit $G(\mathbf{O})t^{-\lambda}G(\mathbf{O})/G(\mathbf{O})$ is an affine bundle over G/P and we let $p : G/P \rightarrow G(\mathbf{O})t^{-\lambda}G(\mathbf{O})/G(\mathbf{O})$ denote the zero-section. Our cohomological theorem states that

$$(1) \quad p_*([\Pi_y^x]) = q^*(\xi^{\theta(x,y)})$$

where $[\Pi_y^x] \in H_T^*(G/P)$ and $\xi^{\theta(x,y)} \in H_T^*(\widetilde{Fl})$ denote the torus-equivariant cohomology classes of projected Richardson varieties and affine Schubert varieties respectively, and q^* is induced by the composition of the inclusion $\text{Gr}_\lambda \rightarrow \text{Gr}$, with the maps $\text{Gr} \simeq \Omega K \rightarrow LK \rightarrow LK/T_{\mathbb{R}} \simeq \widetilde{Fl}$. Here $K \subset G$ denotes the maximal compact subgroup, $T_{\mathbb{R}} \subset T$ the compact torus, and LK and ΩK are the free loop group and based loop group. We also show that the same formula (1) holds in torus-equivariant K -theory. The proof of our cohomological and K -theoretic comparisons (Sections 5 and 6) relies on the study of equivariant localizations. We utilize the machinery developed by Kostant and Kumar [18] where equivariant localizations of Schubert classes in both finite and infinite-dimensional flag varieties are studied.

In Section 6.3, we use our K -theory comparison to prove a conjecture of Knutson, Lam, and Speyer [16] stating that the affine stable Grothendieck polynomials of [22, 23] represent the classes of the structure sheaves of positroid varieties in the K -theory of the Grassmannian. In Section 6.4, we explain the implications, in the case of a cominuscule G/P , towards the comparison of the quantum K -theory of G/P and the K -homology ring of the affine Grassmannian.

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2. COMBINATORIAL COMPARISON

2.1. Let G be a complex simple algebraic group, that is, a complex connected quasi-simple algebraic group of adjoint type. Let B, B^- be opposite Borel subgroup of G and $T = B \cap B^-$ be a maximal torus. Let Q be the coroot lattice and P be the coweight lattice. We denote by P^+ the set of dominant coweights and $Q^+ = Q \cap P^+$. Let $(\alpha_i)_{i \in S}$

be the set of simple roots determined by (B, T) . Let R (resp. R^+ , R^-) be the set of roots (resp. positive roots, negative roots). We denote by W the Weyl group $N(T)/T$. For $i \in S$, we denote by s_i the simple reflection corresponding to i . For $\alpha \in R$, we let r_α denote the corresponding reflection.

Let $W_a = Q \rtimes W$ be the affine Weyl group and $\widehat{W} = P \rtimes W$ be the extended affine Weyl group. It is known that W_a is a normal subgroup of \widehat{W} and is a Coxeter group with generators s_i (for $i \in \tilde{S} = S \cup \{0\}$). Here s_i (for $i \in S$) generates W and $s_0 = t^{-\theta^\vee} s_\theta$ is a simple affine reflection, where θ is the largest positive root of G .

Following [15], we define the length function on \widehat{W} by

$$(a) \quad \ell(t^\chi w) = \sum_{\alpha \in R^+, w^{-1}(\alpha) \in R^+} |\langle \chi, \alpha \rangle| + \sum_{\alpha \in R^+, w^{-1}(\alpha) \in R^-} |\langle \chi, \alpha \rangle + 1|.$$

For any proper subset J of \tilde{S} , let W_J be the subgroup generated by s_i for $i \in J$ and w_J be the maximal element in W_J . We denote by \widehat{W}^J (resp. ${}^J\widehat{W}$) the set of minimal length representatives in \widehat{W}/W_J (resp. $W_J \backslash \widehat{W}$). For $J, K \subset \tilde{S}$, we simply write $\widehat{W}^J \cap {}^K\widehat{W}$ as ${}^K\widehat{W}^J$. If moreover, $J, K \subset S$, then we write W^J for $W \cap \widehat{W}^J$, KW for $W \cap {}^K\widehat{W}^J$ and ${}^KW^J$ for $W \cap {}^K\widehat{W}^J$.

Let Ω be the subgroup of length-zero elements of \widehat{W} . The Bruhat order on W_a extends naturally to \widehat{W} . Namely, for $w_1, w_2 \in W_a$ and $\tau_1, \tau_2 \in \Omega$, we define $\tau_1 w_1 \leq \tau_2 w_2$ if and only if $\tau_1 = \tau_2$ and $w_1 \leq w_2$ in W_a .

2.2. Now we introduce three operations $*$: $\widehat{W} \times \widehat{W} \rightarrow \widehat{W}$, \triangleright : $\widehat{W} \times \widehat{W} \rightarrow \widehat{W}$ and \triangleleft : $\widehat{W} \times \widehat{W} \rightarrow \widehat{W}$. Here $*$ is the *Demazure*, or *monoidal*, product and following [17] we call \triangleright and \triangleleft the *downwards Demazure products*. They were also considered in [14] and [13] and some properties were also discussed there.

We describe $x * y$, $x \triangleright y$ and $x \triangleleft y$ for $x, y \in \widehat{W}$ as follows. See [12, Lemma 1.4].

(1) The subset $\{uv; u \leq x, v \leq y\}$ contains a unique maximal element, which we denote by $x * y$. Moreover, $x * y = u'y = xv'$ for some $u' \leq x$ and $v' \leq y$ and $\ell(x * y) = \ell(u') + \ell(y) = \ell(x) + \ell(v')$.

(2) The subset $\{uy; u \leq x\}$ contains a unique minimal element which we denote by $x \triangleright y$. Moreover, $x \triangleright y = u''y$ for some $u'' \leq x$ with $\ell(x \triangleright y) = \ell(y) - \ell(u'')$.

(3) The subset $\{xv; v \leq y\}$ contains a unique minimal element which we denote by $x \triangleleft y$. Moreover, $x \triangleleft y = xv''$ for some $v'' \leq y$ with $\ell(x \triangleleft y) = \ell(x) - \ell(v'')$.

Now we list some properties of $*$, \triangleright and \triangleleft .

(4) If $x' \leq x$ and $y' \leq y$, then $x' * y' \leq x * y$. See [13, Corollary 1].

(5) If $x' \leq x$, then $x' \triangleleft y \leq x \triangleleft y$. See [13, Lemma 2].

(6) $z \leq x * y$ if and only if $z \triangleleft y^{-1} \leq x$ if and only if $x^{-1} \triangleright z \leq y$. See [14, Appendix].

(7) If J is a proper subset of \tilde{S} , then $\min(W_J x) = w_J \triangleright x$, $\min(x W_J) = x \triangleleft w_J$, $\max(W_J x) = w_J * x$ and $\max(x W_J) = x * w_J$.

2.3. In the rest of this section, we fix a dominant coweight λ . Set $J = \{i \in S; \langle \lambda, \alpha_i \rangle = 0\}$. Any element in $W t^{-\lambda} W \subset \widehat{W}$ can be written in a unique way as $yt^{-\lambda}x^{-1}$ for $x \in W^J$ and $y \in W$. In this case, $\ell(yt^{-\lambda}x^{-1}) = \ell(t^{-\lambda}) + \ell(y) - \ell(x)$. The maximal element in $W t^{-\lambda} W$ is $w_S t^{-\lambda}$ and the minimal element is $t^{-\lambda} w_J w_S$.

Proposition 2.1. *Let $x, x' \in W^J$ and $y, y' \in W$. Then the following conditions are equivalent:*

- (1) $y't^{-\lambda}(x')^{-1} \leq yt^{-\lambda}x^{-1}$;
- (2) There exists $u \in W_J$ such that $y'u \leq y$ and $xu^{-1} \leq x'$;

(3) There exists $v \in W_J$ such that $y' \leq yv$ and $xv \leq x'$.

Proof. (1) \Rightarrow (2): Since $\ell(yt^{-\lambda}x^{-1}) = \ell(t^{-\lambda}x^{-1}) + \ell(y)$, we have that $yt^{-\lambda}x^{-1} = y * t^{-\lambda}x^{-1}$. By 2.2 (6), $y^{-1} \triangleright (y't^{-\lambda}(x')^{-1}) \leq t^{-\lambda}x^{-1}$. In other words, there exists $z \leq y$ such that $z^{-1}y't^{-\lambda}(x')^{-1} \leq t^{-\lambda}x^{-1}$. Now we have that

$$\max(z^{-1}y'W_J)t^{-\lambda} = \max(z^{-1}y't^{-\lambda}(x')^{-1}W) \leq \max(t^{-\lambda}x^{-1}W) = w_Jt^{-\lambda}.$$

Therefore $\max(z^{-1}y'W_J) \leq w_J$ and $z^{-1}y' \in W_J$. We denote $(y')^{-1}z$ by u . Then $u \in W_J$, $y'u = z \leq y$ and $u^{-1}t^{-\lambda}(x')^{-1} \leq t^{-\lambda}x^{-1}$. We have that

$$u^{-1}t^{-\lambda}(x')^{-1} = (t^{-\lambda}w_Jw_S)(w_Sw_Ju^{-1}(x')^{-1}) \quad \text{and} \quad t^{-\lambda}x^{-1} = (t^{-\lambda}w_Jw_S)(w_Sw_Jx^{-1}).$$

Moreover,

$$\begin{aligned} \ell(u^{-1}t^{-\lambda}(x')^{-1}) &= \ell(t^{-\lambda}) + \ell(u) - \ell(x') \\ &= \ell(t^{-\lambda}w_Jw_S) + \ell(w_Sw_J) + \ell(u) - \ell(x') \\ &= \ell(t^{-\lambda}w_Jw_S) + \ell(w_S) - \ell(w_J) + \ell(u) - \ell(x') \\ &= \ell(t^{-\lambda}w_Jw_S) + \ell(w_S) - \ell(w_Ju^{-1}) - \ell(x') \\ &= \ell(t^{-\lambda}w_Jw_S) + \ell(w_S) - \ell(w_Ju^{-1}(x')^{-1}) \\ &= \ell(t^{-\lambda}w_Jw_S) + \ell(w_Sw_Ju^{-1}(x')^{-1}). \end{aligned}$$

Similarly, $\ell(t^{-\lambda}x^{-1}) = \ell(t^{-\lambda}w_Jw_S) + \ell(w_Sw_Jx^{-1})$.

From $u^{-1}t^{-\lambda}(x')^{-1} \leq t^{-\lambda}x^{-1}$ we deduce that $w_Sw_Ju^{-1}(x')^{-1} \leq w_Sw_Jx^{-1}$. Hence $w_Ju^{-1}(x')^{-1} \geq w_Jx^{-1}$ and $xw_J \leq x'u w_J$. By 2.2 (5),

$$xu^{-1} = (xw_J) \triangleleft (w_Ju^{-1}) \leq (x'u w_J) \triangleleft (w_Ju^{-1}) \leq x'.$$

(2) \Rightarrow (1): We have that $yt^{-\lambda}x^{-1} = y(t^{-\lambda}w_Jw_S)(w_Sw_Jx^{-1})$ and $\ell(yt^{-\lambda}x^{-1}) = \ell(y) + \ell(t^{-\lambda}w_Jw_S) + \ell(w_Sw_Jx^{-1})$.

Since $xu^{-1} \leq x'$, we have that

$$xw_J = (xu^{-1})(uw_J) \leq x' * (uw_J) = x'u w_J.$$

Thus $w_Sw_Jx^{-1} \geq w_Sw_Ju^{-1}(x')^{-1}$. Also we have that $y'u \leq y$. Therefore

$$\begin{aligned} y't^{-\lambda}(x')^{-1} &= (y'u)(t^{-\lambda}w_Jw_S)(w_Sw_Ju^{-1}(x')^{-1}) \leq y(t^{-\lambda}w_Jw_S)(w_Sw_Jx^{-1}) \\ &= yt^{-\lambda}x^{-1}. \end{aligned}$$

(2) \Rightarrow (3): Since $y'u \leq y$, by 2.2 (4) $y' \leq y'u * u^{-1} \leq y * u^{-1}$. In other words, there exists $v \leq u^{-1}$ such that $y' \leq yv$. Notice that $u \in W_J$. Hence $v \in W_J$. Since $x \in W^J$, we also have that $xv \leq xu^{-1} \leq x'$.

(3) \Rightarrow (2): Since $y' \leq yv$, by 2.2 (5) $y' \triangleleft v^{-1} \leq yv \triangleleft v^{-1} \leq y$. In other words, there exists $u \leq v^{-1}$ such that $y'u \leq y$. Notice that $v \in W_J$. Hence $u \in W_J$. Since $x \in W^J$, we also have that $xu^{-1} \leq xv \leq x'$. \square

2.4. Define the partial order \preceq on $W^J \times W$ as follows:

$(x', y') \preceq (x, y)$ if and only if there exists $u \in W_J$ such that $x'u \leq x$ and $y'u \geq y$.

Define $Q_J = \{(x, y) \in W^J \times W; y \leq x\}$. Then (Q_J, \preceq) is a subposet of $(W^J \times W, \preceq)$. We shall show in Appendix that Q_J is the same poset as the one studied in [32, 10].

Following [20], we introduce the admissible set as

$$\text{Adm}(-w_S\lambda) = \{z \in \widehat{W}; z \leq t^{-w\lambda} \text{ for some } w \in W\}.$$

Here $-w_S\lambda$ is the unique dominant coweight in the W -orbit of $-\lambda$.

Now we have the following result.

Theorem 2.2.

(1) The map

$$W^J \times W \rightarrow Wt^{-\lambda}W, \quad (x, y) \mapsto yt^{-\lambda}x^{-1}$$

gives an order-preserving, graded, bijection between the poset $(W^J \times W, \preceq)$ and the poset $(Wt^{-\lambda}W, \leq^{op})$. Here \leq^{op} is the opposite Bruhat order on the extended affine Weyl group \widehat{W} .

(2) Its restriction to Q_J gives an order-preserving, graded, bijection between the posets (Q_J, \preceq) and $(Wt^{-\lambda}W \cap \text{Adm}(-w_S\lambda), \leq^{op})$.

Proof. (1) is just a reformulation of the Proposition 2.1. Now we prove (2).

If $(x, y) \in Q_J$, then $y \leq x$. Hence $yt^{-\lambda}x^{-1} \leq xt^{-\lambda}x^{-1} = t^{-x\lambda}$. So $yt^{-\lambda}x^{-1} \in \text{Adm}(-w_S\lambda)$. On the other hand, if $yt^{-\lambda}x^{-1} \in \text{Adm}(-w_S\lambda)$, then $yt^{-\lambda}x^{-1} \leq t^{-w\lambda} = wt^{-\lambda}w^{-1}$ for some $w \in W^J$. Again by Proposition 2.1, there exists $u \in W_J$ such that $y \leq wu \leq x$. Therefore $(x, y) \in Q_J$. \square

Theorem 2.2(2) generalizes [16, Theorem 3.16].

3. APPLICATIONS

3.1. It is a classical result of Björner and Wachs [2] that intervals in the Bruhat order of a Coxeter group satisfy nice combinatorial properties known as *thinness* and *shellability*. Verma [34] showed that the same intervals are *Eulerian*. Dyer [8] extended these results by showing that these intervals and their duals were more generally *EL-shellable*.

Since Theorem 2.2 identifies each Q_J with a convex subposet of (dual) affine Bruhat order we immediately obtain

Corollary 3.1. *The poset Q_J is thin, Eulerian, and EL-shellable.*

This result was first established by Williams [35], who proved the more general result that the poset obtained from Q_J by adjoining a maximal element is shellable.

3.2. Now we discuss the length-generating function $F_\lambda(q)$ of the admissible set $\text{Adm}(-w_S\lambda)$ for a cominuscule coweight λ . By definition,

$$F_\lambda(q) = \sum_{w \in \text{Adm}(-w_S\lambda)} q^{\ell(w)}.$$

This is the number of rational points of the union of all opposite affine Schubert cells corresponding to the admissible set. It is proved in [29] and [37] that this union is the special fiber of the local model of Shimura variety.

Now by Theorem 2.2, $F_\lambda(q) = \sum_{(x,y) \in Q_J} q^{\ell(yt^{-\lambda}x)} = \sum_{(x,y) \in Q_J} q^{\langle \lambda, 2\rho \rangle + \ell(y) - \ell(x)}$, where ρ is the half sum of the positive roots of G .

On the other hand, as we'll see in the appendix, (Q_J, \preceq) is combinatorially equivalent to the poset of totally nonnegative cells in the cominuscule Grassmannian G/P_J . The dimension of the cell corresponding to $(x, y) \in Q_J$ is $\ell(x) - \ell(y)$. Let

$$A_J(q) = \sum_{(x,y) \in Q_J} q^{\ell(x) - \ell(y)}$$

be the rank generating function of totally nonnegative cells in G/P_J . Then we have that

$$(2) \quad F_\lambda(q) = q^{\langle \lambda, 2\rho \rangle} A_J(q^{-1}).$$

In particular, the cardinality of $\text{Adm}(-w_S\lambda)$ is $F_\lambda(1) = A_J(1)$. When G is of classical type, the numbers $A_J(1)$ and in some cases also the generating function $A_J(q)$ have been calculated:

3.2.1. *Type A.* Let $A_{k,n}(q) = A_J(q)$ where G/P_J is the Grassmannian $\text{Gr}(k, n)$ of k -planes in n -space, and similarly define $F_{k,n}(q)$. Postnikov [30] calculated $A_{k,n}(1)$ and Williams [36] established the formula

$$A_{k,n}(q) = q^{-k^2} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (q^{ki} [k-i]^i [k-i+1]^{n-i} - q^{(k+1)i} [k-i-1]^i [k-i]^{n-i})$$

which by (2) gives

$$F_{k,n}(q) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (q^{i(n-k+1)} [k-i]^i [k-i+1]^{n-i} - q^{n+ni-ki} [k-i-1]^i [k-i]^{n-i}).$$

Here $[i] = 1 + q + \cdots + q^{i-1}$ denotes the q -analog of i .

In particular,

$$F_{k,n}(1) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} ((k-i)^i (k-i+1)^{n-i} - (k-i-1)^i (k-i)^{n-i}).$$

The formulas for $F_{1,n}(1)$ and $F_{2,n}(1)$ was first established by Haines in [11, Proposition 8.2 (1) & (2)].

3.2.2. *Type B.* Let $F_{B_n}(q)$ denote $F_\lambda(q)$ for $\lambda = \omega_1^\vee$ the unique cominuscle coweight when G is simple of type B . Similarly define $A_{B_n}(q)$.

Proposition 3.2. *We have*

$$\sum_{n \geq 0} F_{B_n}(q) x^n = \frac{1 + (-q - 3q^2)x + (-q + 5q^3 + 4q^4)x^2 + q^4(-2 - 5q - 3q^2)x^3 + q^6[2]^2x^4}{(1 - q^2x)(1 - (q + q^2)x)(1 - [2]^2x + q^3[2]x^2)}.$$

Proof. Using the combinatorial description in [26, Section 9], we have the recursion

$$A_{B_{n+1}}(q) = 1 + (1 + q)A_{B_n}(q) + \hat{b}_{n+1}(q)$$

for $n \geq 1$, where as in [26, Proposition 11.1], $\hat{b}_n(q) = \sum_{(w_S w_J, y) \in Q_J} q^{\ell(w_S w_J) - \ell(y)}$. This gives

$$\sum_{n \geq 0} A_{B_n}(q) x^n = \frac{\hat{b}(x, q) - (1 + q)x + \frac{x}{1-x}}{1 - (1 + q)x}$$

where $\hat{b}(x, q) = \sum_{n \geq 0} \hat{b}_n(q) x^n$, and we have used the initial conditions $A_{B_0}(q) = 1$ and $A_{B_1}(q) = 2 + q$. Substituting the generating function for $\hat{b}_n(q)$ given in [26, Proposition 11.1], and using (2) gives the stated result. \square

3.2.3. *Type C.* Let $F_{C_n}(q)$ denote $F_\lambda(q)$ where $\lambda = \omega_n^\vee$ is the unique cominuscle coweight when G is simple of type C . Haines [11, Proposition 8.2 (3)] showed that $F_{C_n}(1) = \sum_{i=0}^n 2^{n-i} n! / i!$, which is the greatest integer less than $2^n n! \sqrt{e}$. This calculation was also done by Lam and Williams [26, Proposition 11.3] where it is shown that $F_{C_n}(1)$ satisfies the recurrence $F_{C_0}(1) = 1$ and $F_{C_{n+1}}(1) = 2(n+1)F_{C_n}(1) + 1$.

3.2.4. *Type D.* Let $F_{D_n}(q)$ denote $F_\lambda(q)$ where $\lambda = \omega_1^\vee$ is the cominuscle coweight for G simple of type D , such that G/P_J is an even dimensional quadric. Similarly define $A_{D_n}(q)$.

Proposition 3.3. *We have*

$$\sum_{n \geq 0} F_{D_n}(q)x^n = \frac{1}{(1 - q^2x)(1 - (q + q^2)x)(1 - [2]^2x + q^3[2]x^2)} \times$$

$$1 - (q + 3q^2)x - (q^2 - q^3 - 4q^4)x^2 - (2q + 3q^2 - 2q^3 - 8q^4 - 2q^5 + 3q^6)x^3 + (2q^3 + 3q^4 - 3q^5 - 9q^6 - 4q^7 + q^8)x^4 - (q^6 - 3q^8 - 2q^9)x^5$$

Proof. Using the combinatorial description in [26, Section 9], we have the recursion

$$A_{D_{n+1}}(q) = 1 + (1 + q)A_{D_n}(q) + \hat{d}_{n+1}(q)$$

for $n \geq 2$, where as in [26, Proposition 11.2], $\hat{d}_n(q) = \sum_{(w_S w_J, y) \in Q_J} q^{\ell(w_S w_J) - \ell(y)}$. Declaring the initial conditions $A_{D_0}(q) = 1$, $A_{D_1}(q) = 2 + q$, and $A_{D_2}(q) = 4 + 4q + q^2$ and proceeding as in the proof of Proposition 3.2, we obtain the stated result. \square

3.2.5. *Type D.* The other cominuscle coweights for type D give the even orthogonal Grassmannians. The authors do not know of a calculation of $F_\lambda(q)$ in this case. Part of the enumeration is done in [26, Theorem 11.11].

4. GEOMETRIC COMPARISON

4.1. In this section, we explain some geometry behind the combinatorial comparison. Here we consider three stratified spaces.

Let $J \subset S$. Let P_J be the standard parabolic subgroup of type J and L_J the standard Levi subgroup. Let \mathcal{P}_J be the variety of parabolic subgroups conjugate to P_J . Then it is known that $\mathcal{P}_J \cong G/P_J$. For any $P \in \mathcal{P}_J$, let U_P be the unipotent radical of P .

The first stratified space we consider is the partial flag variety $\mathcal{P}_J \cong G/P_J$. By [27] and [32], $G/P_J = \sqcup_{(x,y) \in Q_J} \Pi_y^x$ where $\Pi_y^x = \pi(X_y^x)$ are the projected Richardson varieties. For any $(x, y) \in Q_J$, the closure of Π_y^x is the union of $\Pi_{y'}^{x'}$, where (x', y') runs over elements in Q_J such that $(x', y') \preceq (x, y)$.

The second stratified space we consider is the variety Z_J introduced by Lusztig in [28]. By definition,

$$Z_J = \{(P, gU_P); P \in \mathcal{P}_J, g \in G\}.$$

This is a variety with a $G \times G$ -action defined by

$$(g, g') \cdot (P, hU_P) = (g'P(g')^{-1}, gh(g')^{-1}U_{g'P(g')^{-1}}).$$

The action of $G \times G$ on Z_J is transitive and the isotropy group of (P_J, U_{P_J}) is $R_J = \{(lu, lu'); l \in L_J, u, u' \in U_{P_J}\}$. We have that $Z_J \cong (G \times G)/R_J$.

For $(x, y) \in W^J \times W$, we define

$$[J, x, y] = (B \times B)(x, y)R_J/R_J,$$

$$[J, x, y]^{+, -} = (B \times B^-)(x, y)R_J/R_J \subset Z_J.$$

Then $[J, x, y]^{+, -} = (1, w_S)[J, x, w_S y]$. By [14, 2.2 & 2.4], $Z_J = \sqcup_{(x,y) \in W^J \times W} [J, x, y]^{+, -}$ and the closure of $[J, x, y]^{+, -}$ in Z_J is the union of $[J, x', y']^{+, -}$, where (x', y') runs over elements in $W^J \times W$ such that $(x', y') \preceq (x, y)$.

The third stratified space is contained in the loop group $G(\mathbf{K})$. Let $\mathbf{O} = \mathbb{C}[[t]]$ and $\mathbf{O}^- = \mathbb{C}[t^{-1}]$. Let I be the inverse image of B under the map $p : G(\mathbf{O}) \rightarrow G$ by sending t to 0 and I^- be the inverse image of B^- under the map $p^- : G(\mathbf{O}^-) \rightarrow G$ by sending t^{-1} to 0. Then we have that $G(\mathbf{O}^-)t^{-\lambda}G(\mathbf{O}) = \sqcup_{w \in W} t^{-\lambda} w I^- w I$. The closure of $I^- w I$ in

$G(\mathbf{O}^-)t^{-\lambda}G(\mathbf{O})$ is the union of $I^-w'I$, where w' runs over elements in $Wt^{-\lambda}W$ such that $w \leq w'$ for the Bruhat order on \widehat{W} .

The three stratified spaces are related by the following diagram.

$$(*) \quad G/P_J \xrightarrow{\iota} Z_J \xleftarrow{f} G(\mathbf{O}^-)t^{-\lambda}G(\mathbf{O}).$$

Here ι is an embedding induced from the map $G \rightarrow Z_J$, $g \mapsto (g, g)R_J/R_J$ and

$$f(gt^{-\lambda}(g')^{-1}) = (p^-(g'), p(g))R_J/R_J$$

for $g \in G(\mathbf{O}^-)$ and $g' \in G(\mathbf{O})$.

Lemma 4.1. *The map f is well-defined.*

Proof. Let $I_1 = \ker(p)$ and $I_1^- = \ker(p^-)$. Let $g, g' \in G$. Suppose that $g't^{-\lambda}g^{-1} \in I_1^-t^{-\lambda}I_1$, we'll show that $(g, g') \in R_J$.

We have that $\emptyset \neq t^\lambda I_1^-g't^{-\lambda} \cap I_1g \subset t^\lambda K^-t^{-\lambda} \cap K$. We shall study this intersection in more detail. We show that

$$(a) \quad K^-t^{-\lambda}I = \cup_{w \in W} I^-wt^{-\lambda}I.$$

Since $I^-w \subset K^-$ for all $w \in W$, $\cup_{w \in W} I^-wt^{-\lambda}I \subset K^-t^{-\lambda}I$. On the other hand, for any $i \in S$ and $w \in W$, $s_i I^-wt^{-\lambda} \subset I^-s_i wt^{-\lambda}I \cup I^-wt^{-\lambda}I$. Hence $s_i \cup_{w \in W} I^-wt^{-\lambda}I \subset \cup_{w \in W} I^-wt^{-\lambda}I$. Since K^- is generated by I^- and s_i for $i \in S$, we have that $K^- \cup_{w \in W} I^-wt^{-\lambda}I = \cup_{w \in W} I^-wt^{-\lambda}I$. In particular, $K^-t^{-\lambda}I = \cup_{w \in W} I^-wt^{-\lambda}I$. (a) is proved.

Similarly,

$$(b) \quad \text{For any } w \in W_J, I^-t^{-\lambda}IwI \subset \cup_{v \in W_J} I^-vt^{-\lambda}I.$$

Now for any $v \in W_J$ and $w \in {}^JW$, we have that $\ell(vt^{-\lambda}w) = \ell(vt^{-\lambda}) - \ell(w)$. Hence $vt^{-\lambda}(I \cap wI^{-w^{-1}}) \subset I^-vt^{-\lambda}$ and $I^-vt^{-\lambda}IwI = I^-vt^{-\lambda}(I \cap wI^{-w^{-1}})wI = I^-vt^{-\lambda}wI$.

Now suppose that $g \in BwB$ for some $w \in W$. Then we may write w as $w = xy$ for $x \in W_J$ and $y \in {}^JW$. Then applying (a) and (b) we deduce that $t^{-\lambda}I_1g \subset \cup_{v \in W} I^-vt^{-\lambda}yI$ and $I^-g't^{-\lambda} \subset \cup_{v \in W} I^-vt^{-\lambda}I$. Since $t^{-\lambda}I_1g \cap I^-g't^{-\lambda} \neq \emptyset$, by the disjointness of the Birkhoff factorization (see [21, Theorem 5.23(g)]) we have $y = 1$ and $g \in P_J$.

Assume that $g = ul$ with $u \in U_{P_J}$ and $l \in L_J$. Then $t^{-\lambda}g^{-1}t^\lambda \subset l^{-1}I_1^-$ and $t^\lambda I_1^-g't^{-\lambda}g^{-1} \subset t^\lambda I_1^-g'l^{-1}I_1^-t^{-\lambda} = t^\lambda I_1^-g'l^{-1}t^{-\lambda}$, where for the last equality we use the fact that G normalizes I_1 . Hence $t^\lambda I_1^-g'l^{-1}t^{-\lambda} \cap I_1 \neq \emptyset$. Now it follows from [21, 5.2.3 (11)] that $I_1 = (I_1 \cap t^\lambda I^-t^{-\lambda})(I_1 \cap t^\lambda I t^{-\lambda})$. Since U normalizes I_1 , comparing Lie algebras and using the fact that I_1 is connected we obtain $(I_1 \cap t^\lambda I t^{-\lambda}) = (I_1 \cap t^\lambda I_1 t^{-\lambda})(I_1 \cap t^\lambda U t^{-\lambda})$ and similarly $(I_1 \cap t^\lambda I^-t^{-\lambda}) = (I_1 \cap t^\lambda I_1^-t^{-\lambda})(I_1 \cap t^\lambda U^-t^{-\lambda})$.

It is easy to see that $t^\lambda U t^{-\lambda} \cap I_1 = \{1\}$ and $t^\lambda U^-t^{-\lambda} \cap I_1 = t^\lambda U_{P_J^-}t^{-\lambda}$. Thus $I_1 = (I_1 \cap t^\lambda I_1^-t^{-\lambda})t^\lambda U_{P_J^-}t^{-\lambda}(I_1 \cap t^\lambda I_1 t^{-\lambda})$. Hence $g'l^{-1} \in U_{P_J^-}$ and $(g, g') \in R_J$. The Lemma is proved. \square

The diagram $*$ is compatible with the respective stratifications: for $(x, y) \in W^J \times W$, $f(I^-yt^{-\lambda}x^{-1}I) = (p(I)x, p^-(I^-)y)R_J/R_J = [J, x, y]^{+, -}$, agreeing with Theorem 2.2(1). The following proposition shows that the map ι preserves the stratifications on G/P_J and Z_J , agreeing with Theorem 2.2(2).

Proposition 4.2. *For $(x, y) \in W^J \times W$, $\iota(G/P_J) \cap [J, x, y]^{+, -} \neq \emptyset$ if and only if $(x, y) \in Q_J$. In this case, $\iota(G/P_J)$ and $[J, x, y]^{+, -}$ intersect transversally and the intersection is $\iota(\Pi_y^x)$.*

Proof. If $g \in BxB \cap B^-yB$, then $(g, g)R_J/R_J \in [J, x, y]^{+, -}$. Thus $\iota(\Pi_y^x) = \iota(\pi(X_y^x)) \subset \iota(G/P_J) \cap [J, x, y]^{+, -}$, where $\pi : G/B \rightarrow G/P_J$ is the projection map. Since $\iota(G/P_J) =$

$\sqcup_{(x,y) \in W^J \times W} \iota(G/P_J) \cap [J, x, y]^{+,-}$ and $G/P_J = \sqcup_{(x,y) \in Q_J} \Pi_y^x$, we have that

$$\iota(G/P_J) \cap [J, x, y]^{+,-} = \begin{cases} \iota(G/P_J), & \text{if } (x, y) \in Q_J; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $\iota(G/P_J)$ is a G_{diag} -orbit on Z_J and $[J, x, y]^{+,-}$ is a $B^- \times B$ -orbit on Z_J and $\text{Lie}(G_{\text{diag}}) + \text{Lie}(B^- \times B) = \text{Lie}(G \times G)$, the intersection is transversal. \square

5. COHOMOLOGICAL COMPARISON

5.1. In this subsection, let \mathcal{G}/\mathcal{B} be a Kac-Moody flag variety [21]. This is an ind-finite ind-scheme with a stratification by finite-dimensional Schubert varieties. Let W denote the Kac-Moody Weyl group with positive (resp. negative) roots R^+ (resp. R^-). We consider cohomology with integer coefficients. Let $\{\xi^v \mid v \in W\}$ denote the torus-equivariant Schubert basis $\xi^v \in H_T^*(\mathcal{G}/\mathcal{B})$ constructed by Kostant and Kumar [18]. Here $T \subset \mathcal{G}$ is the maximal torus of the Kac-Moody group. For $v, w \in W$, we let $d_{v,w} = \xi^v(w) := \xi^v|_w \in H_T^*(\text{pt})$ denote the equivariant localization at the T -fixed point $v \in \mathcal{G}/\mathcal{B}$. A cohomology class $\xi \in H_T^*(\mathcal{G}/\mathcal{B})$ is completely determined by its equivariant localizations. We review certain facts concerning $d_{v,w}$ from [21].

If W is a finite Weyl group, we denote by $w \mapsto w^*$ the conjugation action $w \mapsto w_S w w_S$ by the longest element w_S .

Theorem 5.1. [21, 11.1.11] *Let $v, w \in W$ and $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression. For $1 \leq j \leq p$, set $\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$. Then*

$$d_{v,w} = \sum \beta_{j_1} \cdots \beta_{j_m},$$

where the summation runs over all those $1 \leq j_1 < \cdots < j_m \leq p$ such that $s_{i_{j_1}} \cdots s_{i_{j_m}}$ is a reduced expression of v .

Now we prove some properties of $d_{v,w}$.

Lemma 5.2. *Let W be any Kac-Moody Weyl group.*

(1) *We have*

$$d_{x,x} = \prod_{\alpha \in R^+ \cap xR^-} \alpha.$$

(2) *Suppose W is a finite Weyl group. Then*

$$d_{(x^*)^{-1}, (y^*)^{-1}} = w_S y^{-1} d_{x,y}.$$

(3) *Suppose $x, u, v \in W$ and $\ell(uv) = \ell(u) + \ell(v)$. Then*

$$d_{x,uv} = \sum d_{u',u}(u d_{v',v}),$$

where the summation runs over $u', v' \in W$ such that $x = u'v'$ and $\ell(x) = \ell(u') + \ell(v')$.

(4) *Suppose $x \in {}^J W^K$, $u_1, u_2 \in W_J$, $v_1, v_2 \in W_K$ are such that $xv_1, xv_2 \in {}^J W$, then*

$$d_{u_1 x v_1, u_2 x v_2} = \sum_{\substack{w \in W_J \cap x W_K x^{-1} \\ \ell(u_1 w) = \ell(u_1) - \ell(w)}} d_{u_1 w, u_2} (u_2 d_{x,x}) (u_2 x d_{x^{-1} w^{-1} x v_1, v_2}).$$

Proof. (1) Let $x = s_{i_1} \cdots s_{i_p}$ be a reduced expression. Then

$$d_{x,x} = \beta_{j_1} \cdots \beta_{j_p} = \prod_{\alpha \in R^+ \cap xR^-} \alpha.$$

(2) Let $y = s_{i_1} \cdots s_{i_p}$ be a reduced expression. Then $(y^*)^{-1} = s_{i_p}^* \cdots s_{i_1}^*$ is also a reduced expression. Moreover, $s_{i_{j_1}} \cdots s_{i_{j_m}}$ is a reduced expression of x if and only if $s_{i_{j_m}}^* \cdots s_{i_{j_1}}^*$ is a reduced expression of $(x^*)^{-1}$. For any j , we have that $w_S y^{-1} \beta_j = -w_S s_{i_p} \cdots s_{i_{j+1}} \alpha_{i_j} = s_{i_p}^* \cdots s_{i_{j+1}}^* (-w_S \alpha_{i_j}) = s_{i_p}^* \cdots s_{i_{j+1}}^* \alpha_{i_j}^*$. Now

$$d_{(x^*)^{-1}, (y^*)^{-1}} = \sum (w_S y^{-1} \beta_{j_1}) \cdots (w_S y^{-1} \beta_{j_m}) = w_S y^{-1} d_{x,y}.$$

Here the summation runs over all those $1 \leq j_1 < \cdots < j_m \leq p$ such that $s_{i_{j_1}} \cdots s_{i_{j_m}}$ is a reduced expression of x .

(3) Let $u = s_{i_1} \cdots s_{i_p}$ and $v = s_{i_{p+1}} \cdots s_{i_q}$ be reduced expressions. Let $1 \leq j_1 < \cdots < j_m \leq p < j_{m+1} < \cdots < j_n \leq p$ such that $s_{i_{j_1}} \cdots s_{i_{j_n}}$ be a reduced expression of x . Then

$$\begin{aligned} \beta_{j_1} \cdots \beta_{j_n} &= (\beta_{j_1} \cdots \beta_{j_m})(\beta_{j_{m+1}} \cdots \beta_{j_n}) \\ &= (\beta_{j_1} \cdots \beta_{j_m}) u((s_{i_{p+1}} \cdots s_{i_{j_{m+1}-1}} \alpha_{i_{j_{m+1}}}) \cdots (s_{i_{p+1}} \cdots s_{i_{j_n-1}} \alpha_{i_{j_n}})) \end{aligned}$$

Now part (3) follows from the above theorem.

(4) By part (3),

$$d_{u_1 x v_1, u_2 x v_2} = \sum d_{w_1, u_2} (u_2 d_{w_2, x}) (u_2 x d_{w_3, v_2}),$$

where the summation runs over w_1, w_2, w_3 such that $u_1 x v_1 = w_1 w_2 w_3$ and $\ell(u_1 x v_1) = \ell(w_1) + \ell(w_2) + \ell(w_3)$.

We may also assume that $d_{w_1, u_2}, d_{w_2, x}, d_{w_3, v_2}$ are nonzero. This implies that $w_1 \leq u_2, w_2 \leq x, w_3 \leq v_2$.

Now we study the above conditions on w_1, w_2, w_3 .

We have that $w_1 \in W_J$ and $w_3 \in W_K$. Thus $w_2 \in W_J x W_K$. Since $x = \min(W_J x W_K)$ and $w_2 \leq x$, we must have that $w_2 = x$. Set $w = u_1^{-1} w_1 \in W_J$. Then $x v_1 = w x w_3$ and $w = x(v_1 w_3^{-1}) x^{-1} \in x W_K x^{-1}$. Since $x v_1 \in {}^J W$, we have that

$$\begin{aligned} \ell(w_1 x w_3) - \ell(w_1) &= \ell(x) + \ell(w_3) = \ell(x w_3) = \ell(w^{-1} x v_1) = \ell(w) + \ell(x v_1) \\ &= \ell(w) + \ell(u_1 x v_1) - \ell(u_1). \end{aligned}$$

Hence $\ell(w_1) = \ell(u_1) - \ell(w)$.

On the other hand, for $w \in W_J \cap x W_K x^{-1}$ with $\ell(u_1 w) = \ell(u_1) - \ell(w)$, we set $w_1 = u_1 w$ and $w_3 = x^{-1} w^{-1} x v_1$. Then $w_1 \leq u_1$ and $w_3 \in W_K$. We have that $w_1 x w_3 = u_1 w x x^{-1} w^{-1} x v_1 = u_1 x v_1$. Also

$$\begin{aligned} \ell(u_1 x v_1) &= \ell(w_1 x w_3) \leq \ell(w_1) + \ell(x w_3) = \ell(u_1) - \ell(w) + \ell(w^{-1} x v_1) \\ &\leq \ell(u_1) - \ell(w) + \ell(w) + \ell(x) + \ell(v_1) = \ell(u_1) + \ell(x) + \ell(v_1) \\ &= \ell(u_1 x v_1). \end{aligned}$$

Hence $\ell(u_1 x v_1) = \ell(w_1) + \ell(x w_3) = \ell(w_1) + \ell(x) + \ell(w_3)$.

Part (4) is now proved. □

Lemma 5.3. *Let W be a finite Weyl group, and $u \leq u' \in W$. Then*

$$\sum_{\substack{v \in W \\ u \leq v \leq u'}} d_{u^{-1}, v^{-1}} (v^{-1} w_S d_{w_S u', w_S v}) (v^{-1} w_S d_{w_S, w_S})^{-1} = \delta_{u, u'}.$$

Proof. By [21, 11.1.22], we have

$$\begin{aligned} (v^{-1}w_S d_{w_S u', w_S v})(v^{-1}w_S d_{w_S, w_S})^{-1} &= v^{-1}w_S(d_{w_S u', w_S v} d_{w_S, w_S}^{-1}) \\ &= v^{-1}w_S(w_S v c_{(u')^{-1}, v^{-1}}) \\ &= c_{(u')^{-1}, v^{-1}}. \end{aligned}$$

Here $c_{(u')^{-1}, v^{-1}}$ is defined in [21, 11.1.2].

By [21, 11.1.7(a)], we have

$$LHS = \sum_{\substack{v \in W \\ u \leq v \leq u'}} d_{u^{-1}, v^{-1} c_{(u')^{-1}, v^{-1}}} = \delta_{u, u'}.$$

□

5.2. We shall apply the results of 5.1 in the case where \mathcal{G}/\mathcal{B} is the finite flag variety G/B , and in the case where \mathcal{G}/\mathcal{B} is the affine flag variety $\widetilde{Fl} = G(\mathbf{K})/I$. We return to the conventions of Section 2: W denotes the finite Weyl group and \widehat{W} denotes the extended affine Weyl group. Note that the results of 5.1 from [21] are stated for W_a (that is, for the affine flag variety of the simply-connected G), but extend without change to the extended affine Weyl group \widehat{W} . We shall also abuse notation in two ways. First, if $x, y \in W$, we shall write $d_{x,y}$ without specifying whether we are considering equivariant localizations of affine or finite Schubert basis, since the two agree. Secondly, if $x, y \in \widehat{W}$, then $\xi^x(y)$ normally takes values in $H_T^*(\text{pt})$ for the affine torus \hat{T} . In the following we still denote by $d_{x,y}$ the image of this value in $H_T^*(\text{pt})$ for the finite torus T . That is, we replace each affine root by its classical root.

Let $\text{Gr} = G(\mathbf{K})/G(\mathbf{O})$ denote the affine Grassmannian of G . Let

$$\text{Gr}_\lambda = \overline{G(\mathbf{O})t^{-\lambda}G(\mathbf{O})/G(\mathbf{O})} \subset \text{Gr}$$

be the closure of the $G(\mathbf{O})$ -orbit inside the affine Grassmannian containing the T -fixed point labeled by $t^{-\lambda}$. Then $\text{Gr}_\lambda = \sqcup_{\mu \leq \lambda} G(\mathbf{O})t^{-\mu}G(\mathbf{O})/G(\mathbf{O})$, where the union is over dominant coweights in dominance order.

5.3. For general facts concerning equivariant cohomology, we refer the reader to [9]. We now fix W, J, λ , and $y \in W$, $x, w \in W^J$. The projected Richardson varieties $\Pi_y^x \subset G/P$ are labeled by $(x, y) \in Q_J$. We denote by $[\Pi_y^x] \in H_T^*(G/B)$ the torus-equivariant cohomology class, and by $[\Pi_y^x]|_w \in H_T^*(\text{pt})$ the equivariant localization at a fixed point. Write $\pi : G/B \rightarrow G/P$ for the natural projection.

Proposition 5.4. *We have*

$$[\Pi_y^x]|_w = \sum_{v \in W_J} d_{y, wv^{-1}}(wv^{-1}w_S d_{w_S x^{-1}, w_S v w^{-1}})(wv^{-1}w_J d_{w_J, w_J})^{-1}.$$

Proof. Recall that $X_y \subset G/B$ (resp. $X^x \subset G/B$) denotes the Schubert (resp. opposite Schubert) varieties. Let $X_y^x = X^x \cap X_y \subset G/B$ be the Richardson variety. Then

$$[X_y^x]|_u = [X_y]|_u [X^x]|_u = d_{y,u} w_S d_{w_S x, w_S u}$$

since $[X^x]$ is obtained from $[X_{w_S x}]$ by the action of w_S . Now applying the equivariant pushforward $\pi_* : H_T^*(G/B) \rightarrow H_T^*(G/P)$ to $[X_y^x]$ gives

$$[\Pi_y^x]|_w = (\pi_* [X_y^x])|_w = \sum_{u \in wW_J} [X_y^x]|_u e(\nu_u)^{-1}$$

where $e(\nu_u)$ denotes the equivariant Euler class of the tangent space ν_u at $u \in G/B$ to the fiber $\pi^{-1}(w)$. We calculate that

$$e(\nu_u) = \prod_{\alpha \in R^- : ur_\alpha \in ww_J} u\alpha = uw_J d_{w_J, w_J}$$

by Lemma 5.2(1). Finally, we apply Lemma 5.2(2) to get

$$w_S d_{w_S x, w_S u} = uw_S d_{w_S x^{-1}, w_S u^{-1}}.$$

□

The following result follows immediately from Lemma 5.2(4).

Lemma 5.5. *For any $x, w \in W^J$ and $y \in W$, we have*

$$d_{yt^{-\lambda}x^{-1}, wt^{-\lambda}w^{-1}} = \sum_{\substack{u \in W_J \\ \ell(yu) = \ell(y) - \ell(u)}} d_{yu, w}(wd_{t^{-\lambda}w_J w_S, t^{-\lambda}w_J w_S})(wt^{-\lambda}w_J w_S d_{w_S w_J u^{-1}x^{-1}, w_S w_J w^{-1}}).$$

5.4. The affine Grassmannian Gr is weak homotopy-equivalent to the based loop group ΩK , where $K \subset G$ is a maximal compact subgroup. The affine flag variety \widetilde{Fl} is weak homotopy-equivalent to the quotient $LK/T_{\mathbb{R}}$ of the (unbased) loop group by the compact torus. The torus-equivariant composition $\Omega K \rightarrow LK \rightarrow LK/T_{\mathbb{R}}$ induces a pullback map $r^* : H_T^*(\widetilde{Fl}) \rightarrow H_T^*(\text{Gr})$. The T -fixed points of Gr are labeled by the cosets of \widehat{W}/W . Thus the pullback map can be described in terms of equivariant localizations by noting that $r^*(\xi)(t^{-\lambda}W) = \xi(t^{-\lambda})$ for $\xi \in H_T^*(\widetilde{Fl})$ and a dominant coweight λ .

Let $p : G/P \rightarrow \text{Gr}_\lambda$ embed the partial flag variety G/P as the zero section of the affine bundle $G(\mathbf{O})t^{-\lambda}G(\mathbf{O})/G(\mathbf{O}) \subset \text{Gr}_\lambda$. Let $q^* : H_T^*(\widetilde{Fl}) \rightarrow H_T^*(\text{Gr}_\lambda)$ be the composition of r^* with the restriction $H_T^*(\text{Gr}) \rightarrow H_T^*(\text{Gr}_\lambda)$. The following theorem generalizes [16, Theorem 12.8].

Theorem 5.6. *We have $p_*([\Pi_y^x]) = q^*(\xi^{yt^{-\lambda}x^{-1}})$.*

Proof. We have $yt^{-\lambda}x^{-1} \leq t^\mu$ only if $\max(Wyt^{-\lambda}x^{-1}W) \leq \max(Wt^\mu W)$, only if $\mu' \geq \lambda$, where μ' is the dominant coweight in the W -orbit of μ . Thus $q^*(\xi^{yt^{-\lambda}x^{-1}})$ is non-zero only on T -fixed points of the form $wt^{-\lambda}w^{-1} \in G(\mathbf{O})t^{-\lambda}G(\mathbf{O})/G(\mathbf{O}) \subset \text{Gr}_\lambda$, and so it suffices to compare the two sides at each of these fixed points.

By Lemma 5.2(3), we have

$$(3) \quad d_{y, wv^{-1}} = \sum_{\substack{u \leq v \\ \ell(yu) = \ell(y) - \ell(u)}} d_{yu, w}(wd_{u^{-1}, v^{-1}}).$$

and

$$d_{w_S x^{-1}, w_S v w^{-1}} = d_{w_S x^{-1}, (vw_J)^* w_S w_J w^{-1}} = \sum_{\substack{u' \geq v \\ u' \in W_J}} d_{(u' w_J)^*, (vw_J)^*}((vw_J)^* d_{w_S w_J u'^{-1}x^{-1}, w_S w_J w^{-1}}).$$

Thus applying Lemma 5.2(2),

$$(4) \quad v^{-1}w_S d_{(u' w_J)^*, (vw_J)^*} = w_J d_{w_J u'^{-1}, w_J, v^{-1}} = v^{-1}w_J d_{w_J u', w_J v}$$

and

$$(5) \quad wv^{-1}w_S d_{w_S x^{-1}, w_S v w^{-1}} = \sum_{\substack{u' \geq v \\ u' \in W_J}} (wv^{-1}w_J d_{w_J u', w_J v})(w w_J w_S d_{w_S w_J u'^{-1}x^{-1}, w_S w_J w^{-1}}).$$

Substituting (3) and (5) into Proposition 5.4, and applying Lemma 5.3 to

$$\sum_{\substack{v \in W_J \\ u \leq v \leq u'}} (wd_{u^{-1}, v^{-1}})(wv^{-1}w_J d_{w_J u', w_J v})(wv^{-1}w_J d_{w_J, w_J})^{-1}$$

gives

$$[\Pi_u^x]|_w = \sum_{u \in w_J} d_{yu, w}(ww_J w_S d_{w_S w_J u'^{-1} x^{-1}, w_S w_J w^{-1}}).$$

Now the normal space to $G/P \subset \text{Gr}_\lambda$ at $wt^{-\lambda}w^{-1}$ has equivariant Euler class $e(\nu_w)$ equal to the product of the weights of the T -invariant curves joining $wt^{-\lambda}w^{-1}$ to T -fixed points z inside Gr_λ which are outside of G/P . For $w = w_S w_J$, these are all T -fixed points of the form $z = r_\alpha t^{-\lambda w_J w_S} < t^{-\lambda} w_J w_S$. The product of the T -weights is thus $e(\nu_1) = d_{t^{-\lambda} w_J w_S, t^{-\lambda} w_J w_S}$. A similar calculation gives $e(\nu_w) = w d_{t^{-\lambda} w_J w_S, t^{-\lambda} w_J w_S}$.

Combining with Lemma 5.5, we get

$$p_*([\Pi_u^x])|_{wt^{-\lambda}w^{-1}} = e(\nu_w)[\Pi_u^x]|_w = q^*(\xi^{yt^{-\lambda}x^{-1}})|_w.$$

□

6. K -THEORY COMPARISON

6.1. Kostant and Kumar [19] extended the results of Section 5.1 to the torus-equivariant K -theory $K_T^*(\mathcal{G}/\mathcal{B})$ of a Kac-Moody flag variety. We shall follow the notations of [23], which differ slightly from [19]. Let $\{\psi^v \mid v \in W\}$ denote the basis of $K_T^*(\mathcal{G}/\mathcal{B})$ [23], and define $e_{v,w} = \psi^v(w) := \psi^v|_w \in K_T^*(\text{pt})$. The following result is [23, Proposition 2.10].

Theorem 6.1. *Let $v, w \in W$ and $w = s_{i_1} \cdots s_{i_p}$ be a reduced expression. For $1 \leq j \leq p$, set $\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}$. Then*

$$e_{v,w} = \sum (-1)^{p-m} (1 - e^{\beta_{j_1}}) \cdots (1 - e^{\beta_{j_m}}),$$

where the summation runs over all those $1 \leq j_1 < \cdots < j_m \leq p$ such that $v = s_{i_{j_1}} * \cdots * s_{i_{j_m}}$.

Define E to be the (infinite) matrix $E = (e_{v,w})$, and set $B = (E^{-1})^T$. Define the matrix B' by $b_{u^{-1}, v^{-1}} = v^{-1} w_S b'_{w_S v, w_S u}$. Let M be the Bruhat order matrix given by $m_{v,w} = 1$ if $v \geq w$, and $m_{v,w} = 0$ otherwise. The following result is a variant of [19, Proposition 4.16].

Proposition 6.2. *We have $E^T = DB'M$, where D is the scalar matrix with value $\prod_{\alpha \in R^+} (1 - e^\alpha)$.*

Proof. Define $E_{KK} = (M^T)^{-1}E$ and $B_{KK} = (E_{KK}^{-1})^T$. Let E'_{KK} be the “ E ”-matrix of [19]; then by [23, Appendix A] we have $(e'_{KK})_{v,w} = (e_{KK})_{v^{-1}, w^{-1}}$. Note that $m_{v,w} = m_{v^{-1}, w^{-1}} = m_{w_0 w, w_0 v}$. Proposition 4.16 of [19] gives $E_{KK}^T = DB'_{KK} M^{-1}$, where $(b'_{KK})_{v,u} = v(b_{KK})_{u^{-1} w_S, v^{-1} w_S}$. Then

$$b'_{v,w} = v b_{w^{-1} w_S, v^{-1} w_S} = \sum_{u^{-1} w_S} n_{w^{-1} w_S, u^{-1} w_S} v(b_{KK})_{u^{-1} w_S, v^{-1} w_S} = \sum_u n_{u,v} (b'_{KK})_{v,u},$$

where $N = M^{-1}$. It follows that $E^T = (M^T E_{KK})^T = E_{KK}^T M = DB'_{KK} = DB'M$. □

Now we prove some properties of $e_{v,w}$.

Lemma 6.3. *Let W be any Kac-Moody Weyl group.*

(1) We have

$$e_{x,x} = \prod_{\alpha \in R^+ \cap xR^-} (1 - e^\alpha).$$

(2) Suppose W is a finite Weyl group. Then

$$e_{(x^*)^{-1}, (y^*)^{-1}} = w_S y^{-1} e_{x,y}.$$

(3) Suppose $x, u, v \in W$ and $\ell(uv) = \ell(u) + \ell(v)$. Then

$$e_{x,uv} = \sum e_{u',u}(ue_{v',v}),$$

where the summation runs over $u', v' \in W$ such that $x = u' * v'$.

The proofs are similar to the proof of Lemma 5.2. The analogue of Lemma 5.2 (4) can also be formulated, but the statement is more complicated. We only need the following special case which can be proved easily from Lemma 6.3 (3).

Lemma 6.4. For any $x, w \in W^J$ and $y \in W$, we have

$$e_{yt^{-\lambda}x^{-1}, wt^{-\lambda}w^{-1}} = \sum_{\substack{u \in W_J, y' \in W \\ y = y' * u^{-1}}} e_{y',w}(we_{t^{-\lambda}w_J w_S, t^{-\lambda}w_J w_S})(wt^{-\lambda}w_J w_S e_{w_S w_J u^{-1}x^{-1}, w_S w_J w^{-1}}).$$

Lemma 6.5. Let W be a finite Weyl group, and $u \leq u' \in W$. Then

$$\sum_{\substack{v, u'' \in W \\ u \leq v \leq u'' \leq u'}} e_{u^{-1}, v^{-1}}(v^{-1}w_S e_{w_S u'', w_S v})(v^{-1}w_S e_{w_S, w_S})^{-1} = \delta_{u, u'}.$$

Proof. By Proposition 6.2, we have that

$$\sum_{\substack{u'' \in W \\ v \leq u'' \leq u'}} e_{w_S u'', w_S v} e_{w_S, w_S}^{-1} = b'_{w_S v, w_S u'}.$$

By definition, $b_{u'^{-1}, v^{-1}} = v^{-1}w_S b'_{w_S v, w_S u'}$. Thus

$$\sum_{\substack{v, u'' \in W \\ u \leq v \leq u'' \leq u'}} e_{u^{-1}, v^{-1}}(v^{-1}w_S e_{w_S u'', w_S v})(v^{-1}w_S e_{w_S, w_S})^{-1} = \sum_{\substack{v \in W \\ u \leq v \leq u'}} e_{u^{-1}, v^{-1}} b_{u'^{-1}, v^{-1}} = \delta_{u, u'}.$$

□

6.2. We now establish the analogue of Theorem 5.6 in K -theory. The strategy of the proof is identical, so we shall be briefer in the explanations. In this section, we let $[\mathcal{O}_{\Pi_y^x}] \in K_T^*(G/P)$ denote the class of the structure sheaf of Π_y^x in the torus-equivariant K -theory of G/P .

Theorem 6.6. We have $p_*([\mathcal{O}_{\Pi_y^x}]) = q^*(\psi^{yt^{-\lambda}x^{-1}})$.

Proof. By the proof of Theorem 5.6, $q^*(\psi^{yt^{-\lambda}x^{-1}})$ is non-zero only on T -fixed points of the form $wt^{-\lambda}w^{-1} \in G(\mathbf{O})t^{-\lambda}G(\mathbf{O})/G(\mathbf{O}) \subset \text{Gr}_\lambda$, and so it suffices to compare the two sides at each of these fixed points.

By Lemma 6.3(3), we have

$$(6) \quad e_{y, wv^{-1}} = \sum_{\substack{u \leq v, y' \in W \\ y = y' * u^{-1}}} e_{y', w}(we_{u^{-1}, v^{-1}}).$$

and

$$e_{w_S x^{-1}, w_S v w^{-1}} = e_{w_S x^{-1}, (v w_J)^* w_S w_J w^{-1}} = \sum e_{(u'' w_J)^*, (v w_J)^*} ((v w_J)^* e_{w_S w_J u'^{-1} x^{-1}, w_S w_J w^{-1}}),$$

where the summation runs over all $u', u'' \in W$ such that $(u'' w_J)^* * (w_S w_J u'^{-1} x^{-1}) = w_S x^{-1}$. By definition, $w_S w_J u'^{-1} x^{-1} \in W_{J^*} w_S x^{-1} = w_S W_J x^{-1}$ and $u' \in W_J$. Then $w_S w_J u'^{-1} x^{-1} = (w_J u'^{-1})^* w_S x^{-1}$, here $(w_J u'^{-1})^* \in W_{J^*}$ and $w_S x^{-1}$ is the maximal element in $W_{J^*} w_S x^{-1}$. Hence $(u'' w_J)^* * (w_S w_J u'^{-1} x^{-1}) = w_S x^{-1}$ if and only if $(u'' w_J)^* \geq ((w_J u'^{-1})^*)^{-1}$, i.e., $u'' w_J \geq u' w_J$. This is equivalent to say that $u'' \leq u'$. Hence

$$(7) \quad e_{w_S x^{-1}, w_S v w^{-1}} = \sum_{v \leq u'' \leq u' \text{ in } W_J} e_{(u'' w_J)^*, (v w_J)^*} ((v w_J)^* e_{w_S w_J u'^{-1} x^{-1}, w_S w_J w^{-1}})$$

Thus applying Lemma 6.3(2),

$$(8) \quad v^{-1} w_S e_{(u'' w_J)^*, (v w_J)^*} = w_J e_{w_J u''^{-1}, w_J v^{-1}} = v^{-1} w_J e_{w_J u'', w_J v}$$

and

$$(9) \quad w v^{-1} w_S e_{w_S x^{-1}, w_S v w^{-1}} = \sum_{v \leq u'' \leq u' \text{ in } W_J} (w v^{-1} w_J e_{w_J u'', w_J v}) (w w_J w_S e_{w_S w_J u'^{-1} x^{-1}, w_S w_J w^{-1}}).$$

Similar to Proposition 5.4, we have that

$$(10) \quad [\mathcal{O}_{\Pi_y^x}]|_w = \sum_{v \in W_J} e_{y, w v^{-1}} (w v^{-1} w_S e_{w_S x^{-1}, w_S v w^{-1}}) (w v^{-1} w_J e_{w_J, w_J})^{-1}$$

where we use the fact that in $K_T^*(G/B)$ one has

$$[\mathcal{O}_{X_w}] [\mathcal{O}_{X^u}] = [\mathcal{O}_{X_w^u}]$$

which follows from [3, Lemma 1].

Substituting (6) and (9) into (10), we have that

$$[\mathcal{O}_{\Pi_y^x}]|_w = \sum e_{y', w} (w e_{u^{-1}, v^{-1}}) (w v^{-1} w_J e_{w_J u'', w_J v}) (w v^{-1} w_J e_{w_J, w_J})^{-1} (w w_J w_S e_{w_S w_J u'^{-1} x^{-1}, w_S w_J w^{-1}}),$$

where the summation is over $u \leq v \leq u'' \leq u'$ in W_J and $y' \in W$ such that $y = y' * u^{-1}$.

Applying Lemma 6.5 to

$$\sum_{\substack{v, u'' \in W_J \\ u \leq v \leq u'' \leq u'}} (w e_{u^{-1}, v^{-1}}) (w v^{-1} w_J e_{w_J u'', w_J v}) (w v^{-1} w_J e_{w_J, w_J})^{-1}$$

gives

$$[\mathcal{O}_{\Pi_y^x}]|_w = \sum_{\substack{u \in W_J, y' \in W \\ y = y' * u^{-1}}} e_{y', w} (w w_J w_S e_{w_S w_J u^{-1} x^{-1}, w_S w_J w^{-1}}).$$

Now the normal space to $G/P \subset \text{Gr}_\lambda$ at $wt^{-\lambda} w^{-1}$ has equivariant Euler class $e(\nu_w)$ equal to the product of the weights of the T -invariant curves joining $wt^{-\lambda} w^{-1}$ to T -fixed points z inside Gr_λ which are outside of G/P . For $w = w_S w_J$, these are all T -fixed points of the form $z = r_\alpha t^{-\lambda w_J w_S} < t^{-\lambda} w_J w_S$. The product of the T -weights is thus $e(\nu_1) = e_{t^{-\lambda} w_J w_S, t^{-\lambda} w_J w_S}$. A similar calculation gives $e(\nu_w) = w e_{t^{-\lambda} w_J w_S, t^{-\lambda} w_J w_S}$.

Combining with Lemma 6.4, we get

$$p_*([\mathcal{O}_{\Pi_y^x}]|_{wt^{-\lambda} w^{-1}}) = e(\nu_w) [\mathcal{O}_{\Pi_y^x}]|_w = q^*(\psi^{y t^{-\lambda} x^{-1}})|_w.$$

□

6.3. For background material on the symmetric function notation used in this section, we refer the reader to [4, 23]. The general strategy of this section is similar to [16, Section 7].

In this section we let $G = PGL(n, \mathbb{C})$ and G/P be the Grassmannian $\text{Gr}(k, n)$ of k -planes in \mathbb{C}^n . In [4], Buch defined *stable Grothendieck polynomials* $G_\lambda(X) \in \hat{\Lambda}$ for each partition λ , lying in the graded completion $\hat{\Lambda}$ of the ring of symmetric functions. Buch showed that the K -theory $K^*(\text{Gr}(k, n))$ of the Grassmannian could be presented as $\Gamma/I_{k,n}$, where $\Gamma = \prod_\lambda \mathbb{Z} \cdot G_\lambda(X)$, and $I_{k,n}$ is the ideal spanned (as a direct product) by all G_λ where λ is not contained in a $k \times (n - k)$ rectangle. (Buch considered the direct sum rather than product of the $\mathbb{Z} \cdot G_\lambda(X)$, but the quotient is the same.)

In [22], symmetric functions $\tilde{G}_w(X)$ called *affine stable Grothendieck polynomials* were defined for each element $w \in \widehat{W}$ of the affine Weyl group (in this case, the affine symmetric group). Let $\Lambda^{(n)}$ be the quotient of the ring of symmetric functions by the ideal generated by all monomial symmetric functions m_λ , for $\lambda_1 \geq n$. Let $\hat{\Lambda}^{(n)}$ be the graded completion of $\Lambda^{(n)}$. Let $r^* : K^*(\widehat{Fl}) \rightarrow K^*(\text{Gr})$ denote the pullback map in K -theory, as in Subsection 5.4. In [23], it was shown that $K(\text{Gr}) \simeq \hat{\Lambda}^{(n)}$ and that under this isomorphism one has

$$(11) \quad r^*(\psi^w) = \tilde{G}_w.$$

The following result was conjectured in [16, Conjecture 7.11].

Theorem 6.7. *Let ω_k denote the k -th fundamental coweight. Under the isomorphism $\kappa : K(\text{Gr}(k, n)) \simeq \Gamma/I_{k,n}$, we have*

$$\kappa([\mathcal{O}_{\Pi_y^x}]) = \tilde{G}_{yt^{-\omega_k}x^{-1}}$$

where the right hand side is considered as an element of the quotient $\Gamma/I_{k,n}$.

Proof. When λ is the fundamental coweight ω_k we have $G/P \simeq \text{Gr}_\lambda \subset \text{Gr}$ (see [16, Section 7]). Combining (the non-equivariant image of) Theorem 6.6 with (11), it thus remains to check that the inclusion $\iota : G/P \hookrightarrow \text{Gr}$ induces the natural quotient map $\hat{\Lambda}^{(n)} \rightarrow \Gamma/I_{k,n}$.

The ring $\hat{\Lambda}^{(n)}$ contains distinguished symmetric functions $G_{(m)}(X) = \tilde{G}_{s_{m-1}s_{m-2}\cdots s_0}(X)$ for $1 \leq m < n$. The completion of the subring generated by the $G_{(m)}(X)$ is exactly $\hat{\Lambda}^{(n)}$. The ring homomorphism $\iota^* : K(\text{Gr}) \rightarrow K(\text{Gr}(k, n))$ is compatible with graded completions, and is thus determined by the images of $G_{(m)}(X)$. Now, in $K^*(\text{Gr})$, $\tilde{G}_{s_{m-1}s_{m-2}\cdots s_0}(X)$ represents the pullback $r^*(\psi^{s_{m-1}s_{m-2}\cdots s_0})$. For $m \leq n - k$, modulo length-zero elements of \widehat{W} (one has $\tilde{G}_v(X) = \tilde{G}_u(X)$ if v and u differ by a length-zero element), $s_{m-1}s_{m-2}\cdots s_0$ is the same as $s_{k+m-1}\cdots s_{k+1}s_k t^{-\omega_k} w_J w_S$. But under Buch's isomorphism $K^*(\text{Gr}) \simeq \Gamma/I_{k,n}$, the opposite Schubert variety $\pi(X_{s_{k+m-1}\cdots s_{k+1}s_k}) = \Pi_{s_{k+m-1}\cdots s_{k+1}s_k}^{w_J w_S}$ is represented by the symmetric function $G_{(m)}(X)$ as well [4, Theorem 8.1]. Similarly, if $m > n - k$ one sees that ι^* sends $G_{(m)}(X)$ to 0. Thus ι^* induces the natural map $\hat{\Lambda}^{(n)} \rightarrow \Gamma/I_{k,n}$. \square

6.4. In [24], Lam and Shimozono, following work of Peterson, showed that the quantum cohomology rings $QH^*(G/P)$ of partial flag varieties could, after localization, be identified with a quotient of the homology $H_*(\text{Gr})$ of the affine Grassmannians. In particular, the 3-point Gromov-Witten invariants of G/P could be recovered from the homology Schubert structure constants of $H_*(\text{Gr})$.

Let G/P be a cominuscule flag variety. In this section, we discuss the implications of Theorem 6.6 towards the comparison of the quantum K -theory $QK^*(G/P)$ of G/P and K -homology $K_*(\text{Gr})$ of the affine Grassmannian. We will work in the non-equivariant setting; the T -equivariant statements are analogous. We now define four sets of integers.

- (1) For $u, v, w \in \widehat{W}/W$, let $d_{uv}^w \in \mathbb{Z}$ denote the K -homology Schubert structure constants of $K_*(\text{Gr})$, defined in [23, (5.3)] (we will only consider the non-equivariant structure constants). We remark that in [23] only the affine Grassmannian Gr of a simply-connected simple algebraic group is considered, but the extension is straightforward; see for example [25].
- (2) For $u \in \widehat{W}$, and $y \in \widehat{W}^S$ a minimal length coset representative of \widehat{W}/W we can consider the coefficient k_y^u of the K -cohomology Schubert class ψ_{Gr}^y in $r^*(\psi_{\widetilde{Fl}}^u)$, where $r^* : K^*(\widetilde{Fl}) \rightarrow K^*(\text{Gr})$ denotes the pullback map in K -theory, as in Subsection 5.4.
- (3) For a positroid variety Π_v^u and $y \in W^J$, consider the coefficient $\pi_{(u,v)}^y$ of the (class of the) Schubert structure sheaf $[\mathcal{O}_{X_y}]$ in $[\mathcal{O}_{\Pi_v^u}] \in K^*(G/P)$.
- (4) For $u, v, w \in W^J$, consider the K -theoretic Gromov-Witten invariant $I_d(u, v, w) = I_d(\mathcal{O}_{X_u}, \mathcal{O}_{X_v}, (\mathcal{O}_{X_w})^\vee)$; see for example [5, Section 5]. Here $\{[(\mathcal{O}_{X_w})^\vee]\}$ is the dual basis to $\{[\mathcal{O}_{X_w}]\}$ in $K^*(G/P)$. The K -theoretic Gromov-Witten invariant is defined as the Euler characteristic of the product of the pullbacks of these structure sheaves to the moduli space $M_{d,3}(G/P)$ of degree d , genus zero, rational curves in G/P with three marked points.

We now compare the four sets of integers.

- (1) By [23, (5.1), (5.4) and Theorem 5.4],

$$d_{uv}^w = \sum_{\substack{x \in \widehat{W} \\ x*v=w}} (-1)^{\ell(w)-\ell(v)-\ell(x)} k_u^x.$$

Thus the k_u^x determine the d_{uv}^w , and it is easy to see that (picking v and u appropriately) the d_{uv}^w also determine the k_u^x .

- (2) By the cominuscule assumption we have $G/P \simeq \text{Gr}_\lambda \subset \text{Gr}$, where λ is the appropriate cominuscule coweight. Thus p_* can be identified with the identity, and Theorem 6.6 states that $q^*(\psi^{yt^{-\lambda}x^{-1}}) = [\mathcal{O}_{\Pi_y^x}]$. Now suppose that $x = w_S w_J$ and $y \in W^J$. Then $\Pi_y^x = \pi(X_y)$ is a usual Schubert variety in G/P , and since $yt^{-\lambda}w_J w_S \in \widehat{W}^S$, we have $r^*(\psi_{\widetilde{Fl}}^y) = (\psi_{\text{Gr}}^y)$. It follows that the coefficient $\pi_{(u,v)}^y$ is equal to $k_y^{vt^{-\lambda}u^{-1}}$.
- (3) In [5], Buch, Chaput, Mihalcea, and Perrin studied the geometry of the Gromov-Witten varieties associated to cominuscule G/P . An unpublished consequence of their work, communicated to us by L. Mihalcea, is that

- (12) $I_d(u, v, w)$ is equal to the coefficient of a Schubert structure sheaf $[\mathcal{O}_{X_w}]$ in $[\mathcal{O}_{\Pi_y^x}]$

where Π_y^x is a projected Richardson variety which depends on d, u, v . For an explicit description of Π_y^x in type A see [16, Section 8]. For the classical types the explicit description can presumably be recovered from [6], and for other cominuscule types see [7]. Thus the coefficients $\pi_{(x,z)}^y$ determine all the coefficients $I_d(u, v, w)$.

Corollary 6.8. *Let G/P be cominuscule. Assuming (12), the K -homology Schubert structure constants determine the 3-point K -theoretic Gromov-Witten invariants of G/P .*

APPENDIX

Here we prove that the poset (Q_J, \preceq) is combinatorially equivalent to the poset introduced by Rietsch in [32, Section 5] and by Goodearl and Yakimov in [10, Theorem 1.8].

Following [32], we set

$$Q'_J = \{(a, b, c) \in W_{\max}^J \times W_J \times W^J; a \leq cb\}$$

and define the partial order \leq on Q'_J as follows.

For $(a, b, c), (a', b', c') \in Q'_J$, define

$$(a', b', c') \leq (a, b, c)$$

if there exists $u'_1, u'_2 \in W_J$ with $u'_1 u'_2 = b'$, $\ell(u'_1) + \ell(u'_2) = \ell(b')$ and

$$ab^{-1} \leq a'(u'_2)^{-1} \leq c' u'_1 \leq c.$$

Following [10], we set

$$\Omega_J = \{(a, b) \in W_{\max}^J \times W; a \leq b\}$$

and define the partial order \leq on Ω_J as follows.

For $(a, b), (a', b') \in \Omega_J$, define

$$(a', b') \leq (a, b)$$

if there exists $z \in W_J$ such that $a \leq a'z$ and $b'z \leq b$.

Proposition 6.9. *The maps*

$$\begin{aligned} f : Q'_J &\rightarrow \Omega_J, & (a, b, c) &\mapsto (a, cb) \\ g : \Omega_J &\rightarrow Q'_J, & (a, b) &\mapsto (\min(bW_J), ab^{-1} \min(bW_J)) \\ h : Q_J &\rightarrow Q'_J, & (x, y) &\mapsto (\max(yW_J), y^{-1} \max(yW_J), x) \end{aligned}$$

give order-preserving bijections between the posets (Q'_J, \leq) , (Ω_J, \leq) and (Q_J, \preceq) .

Proof. For $(a, b, c) \in Q'_J$, $f(a, b, c) = (a, cb)$, $g \circ f(a, b, c) = (c, ab^{-1})$ and $h \circ g \circ f(a, b, c) = (a, b, c)$. Similarly, $g \circ f \circ h$ is an identity map on Q_J and $f \circ h \circ g$ is an identity map on Ω_J . Thus f, g, h are all bijective.

Now it suffices to show that f, g, h preserve the partial orders.

Let $(a, b, c), (a', b', c') \in Q'_J$ with $(a', b', c') \leq (a, b, c)$. Then there exists $u'_1 \in W_J$ such that $ab^{-1} \leq a'(b')^{-1}u'_1 \leq c'u'_1 \leq c$. Thus $a = ab^{-1} * b \leq a'(b')^{-1}u'_1 * b$. In other words, there exists $v \leq b$ such that $a \leq a'(b')^{-1}u'_1 v$. Let $z = (b')^{-1}u'_1 v$. Then $c'b'z = c'u'_1 v \leq cv \leq cb$ since $c \in W^J$. Hence $(a', c'b') \leq (a, cb)$ in Ω_J .

Let $(a, b), (a', b') \in \Omega_J$ with $(a', b') \leq (a, b)$. Then there exists $z \in W_J$ with $a \leq a'z$ and $b'z \leq b$. We assume that $g(a, b) = (x, y)$ and $g(a', b') = (x', y')$. Then there exists $u, u' \in W_J$ such that $x = bu$, $y = au$, $x' = b'u'$ and $y' = a'u'$. Since $b'z \leq b$, we have that $b'z \triangleleft u \leq b \triangleleft u \leq bu = x$. Hence there exists $v \leq u$ such that $x'(u')^{-1}zv = b'zv \leq x$.

Since $a \in W_{\max}^J$,

$$y = au \leq av = a \triangleleft v \leq a'z \triangleleft v \leq a'zv = y'(u')^{-1}zv.$$

Thus $(x', y') \preceq (x, y)$ in Q_J .

Now let $(x, y), (x', y') \in Q_J$ with $(x', y') \preceq (x, y)$. Then there exists $u \in W_J$ such that $x'u \leq x$ and $y'u \geq y$. So we have that $y' * u \geq y'u \geq y$. In other words, there exists $v \leq u$ with $y' * u = y'v$ and $\ell(y'v) = \ell(y') + \ell(v)$. Since $x' \in W^J$, we also have that $y'v \leq x'v \leq x'u \leq x$.

We assume that $h(x, y) = (a, b, x)$ and $h(x', y') = (a', b', x')$. Then $y = ab^{-1}$ and $y' = a'(b')^{-1}$. We have that

$$ab^{-1} = y \leq y'v = a'(b')^{-1}v \leq x'v \leq x.$$

Since $a' \in W_{\max}^J$, $\ell(a'(b')^{-1}v) = \ell(a') - \ell((b')^{-1}v)$ and $\ell(y'v) = \ell(y') + \ell(v) = \ell(a') - \ell(b') + \ell(v)$. So $\ell((b')^{-1}v) + \ell(v) = \ell(b')$. Thus $(a', b', x') \leq (a, b, x)$ in Q'_J . \square

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